

SOLVABLE LEIBNIZ ALGEBRAS WITH TRIANGULAR NILRADICALS

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ABSTRACT. In this paper the description of solvable Lie algebras with triangular nilradicals is extended to Leibniz algebras. It is proven that the matrices of the left and right operators on elements of Leibniz algebra have upper triangular forms. We establish that solvable Leibniz algebra of a maximal possible dimension with a given triangular nilradical is a Lie algebra. Furthermore, solvable Leibniz algebras with triangular nilradicals of low dimensions are classified.

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1. INTRODUCTION

Leibniz algebras were introduced at the beginning of the 90s of the past century by J.-L. Loday [3]. They are a generalization of well-known Lie algebras, which admit a remarkable property that an operator of right multiplication is a derivation.

From the classical theory of Lie algebras it is well known that the study of finite-dimensional Lie algebras was reduced to the nilpotent ones [11], [12]. In the Leibniz algebra case there is an analogue of Levi's theorem [4]. Namely, the decomposition of a Leibniz algebra into a semidirect sum of its solvable radical and a semisimple Lie algebra is obtained. The semisimple part can be described from simple Lie ideals (see [5]) and therefore, the main focus is to study the solvable radical.

The analysis of several works devoted to the study of solvable Lie algebras (for example [1, 2, 13, 14, 15], where solvable Lie algebras with various types of nilradical were studied, such as naturally graded filiform and quasi-filiform algebras, abelian, triangular, etc.) shows that we can also apply similar methods to solvable Leibniz algebras with a given nilradical. In fact, any solvable Lie algebra can be represented as an algebraic sum of a nilradical and its complimentary vector space. Mubarakdjanov proposed a method, which claims that the dimension of the complimentary vector space does not exceed the number of nil-independent derivations of the nilradical [12]. Extension of this method to Leibniz algebras is shown in [6]. Usage of this method yields a classification of solvable Leibniz algebras with given nilradicals in [6, 7, 8, 9, 10].

In this article we present the description of solvable Leibniz algebras whose nilradical is a Lie algebra of upper triangular matrices. Since in the work [14] solvable Lie algebras with triangular nilradical are studied, we reduce our study to non-Lie Leibniz algebras.

Recall, that in [14] solvable Lie algebras with triangular nil-radicals of minimum and maximum possible dimensions were described. Moreover, uniqueness of a Lie algebra of maximal possible dimension with a given triangular nilradical is established.

In order to realize the goal of our study we organize the paper as follows. In Section 2 we give the necessary preliminary results. Section 3 is devoted to the description of a finite-dimensional solvable Leibniz algebras with upper triangular nilradical. We establish that such Leibniz algebras of minimum and maximum possible dimensions are Lie algebras. Finally, in Section 4 we present complete description of the results of Section 3 in low dimensions.

Throughout the paper we consider finite-dimensional vector spaces and algebras over the field \mathbb{C} . Moreover, in the multiplication table of an algebra omitted products are assumed to be zero and if it is not stated otherwise, we will consider non-nilpotent solvable algebras.

2. PRELIMINARIES

In this section we give the basic concepts and the results used in the studying of Leibniz algebras with triangular nilradicals.

Definition 2.1. An algebra $(L, [-, -])$ over a field F is called a Leibniz algebra if for any $x, y, z \in L$ the so-called Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

holds.

Every Lie algebra is a Leibniz algebra, but the bracket in the Leibniz algebra does not possess a skew-symmetric property.

Definition 2.2. For a given Leibniz algebra L the sequences of two-sided ideals defined recursively as follows:

$$L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \geq 1, \quad L^{[1]} = L, \quad L^{[s+1]} = [L^{[s]}, L^{[s]}], \quad s \geq 1.$$

are called the lower central and the derived series of L , respectively.

Definition 2.3. A Leibniz algebra L is said to be nilpotent (respectively, solvable), if there exists $n \in \mathbb{N}$ ($m \in \mathbb{N}$) such that $L^n = 0$ (respectively, $L^{[m]} = 0$).

It is easy to see that the sum of any two nilpotent ideals is nilpotent. Therefore the maximal nilpotent ideal always exists.

Definition 2.4. The maximal nilpotent ideal of a Leibniz algebra is said to be a nilradical of the algebra.

Recall, that a linear map $d : L \rightarrow L$ of a Leibniz algebra L is called a derivation if for all $x, y \in L$ the following condition holds:

$$d([x, y]) = [d(x), y] + [x, d(y)].$$

For a given element x of a Leibniz algebra L we consider a right multiplication operators $R_x : L \rightarrow L$ defined by $R_x(y) = [y, x], \forall y \in L$ and the left multiplication operators $L_x : L \rightarrow L$ defined by $L_x(y) = [x, y], \forall y \in L$. It is easy to check that operator R_x is a derivation. This kind of derivations are called *inner derivations*.

Linear maps f_1, \dots, f_k are called *nil-independent*, if

$$\alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_k f_k$$

is not nilpotent for all values α_i , except simultaneously zero.

Let R be a solvable Leibniz algebra with a nilradical N . We denote by Q the complementary vector space of the nilradical N in the algebra R .

Proposition 2.5. [6] Let R be a solvable Leibniz algebra and N its nilradical. Then the dimension of the complementary vector space Q is not greater than the maximal number of nil-independent derivations of N .

Let us consider a finite-dimensional Lie algebra $T(n)$ of upper-triangular matrices with $n \geq 3$ over the field of complex numbers. The products of the basis elements $\{N_{ij} \mid 1 \leq i < j \leq n\}$ of $T(n)$, where N_{ij} is a matrix with the only non-zero entry at i -th row and j -th column equal to 1, can be computed by

$$[N_{ij}, N_{kl}] = \delta_{jk} N_{il} - \delta_{il} N_{kj}.$$

For a natural number f let $G(n, f)$ be a set of solvable Lie algebras of dimension $\frac{1}{2}n(n-1) + f$ with nilradical $T(n)$. Let $Q = \langle X^1, X^2, \dots, X^f \rangle$, where Q is the complementary vector space of the nilradical $T(n)$ to an algebra from $G(n, f)$.

Denote

$$(1) \quad [N_{ij}, X^\alpha] = \sum_{1 \leq q-p < n} a_{ij,pq}^\alpha N_{pq}, \quad [X^\alpha, N_{ij}] = \sum_{1 \leq q-p < n} b_{ij,pq}^\alpha N_{pq}, \quad [X^\alpha, X^\beta] = \sum_{1 \leq q-p < n} \sigma_{pq}^{\alpha\beta} N_{pq},$$

where $1 \leq \alpha, \beta \leq f$ and $a_{ij,pq}^\alpha, b_{ij,pq}^\alpha, \sigma_{pq}^{\alpha\beta} \in \mathbb{C}$, $p < q \leq n$.

Let N be a vector column $(N_{12} \ N_{23} \ \dots \ N_{(n-1)n} \ N_{13} N_{24} \dots \ N_{(n-2)n} \dots \ N_{1n})^T$ then we have

$$R_{X^\alpha}(N) = A^\alpha N, \quad L_{X^\alpha}(N) = B^\alpha N,$$

where $A^\alpha = (a_{ij,pq}^\alpha)$ and $B^\alpha = (b_{ij,pq}^\alpha)$, $1 \leq i < j \leq n$, $1 \leq p < q \leq n$ are $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ complex matrices.

The following lemma provides some information about the structure matrices above.

Lemma 2.6. [14] *The structure matrices $A^\alpha = (a_{ij,pq}^\alpha)$, $1 \leq i < j \leq n$, $1 \leq p < q \leq n$ have the following properties:*

- (i) *They are upper triangular;*
- (ii) *The only off-diagonal matrix elements that do not vanish identically and cannot be annuled by a redefinition of the elements X^α are:*

$$a_{12,2n}^\alpha, \quad a_{i(i+1),1n}^\alpha \quad (2 \leq i \leq n-2), \quad a_{(n-1)n,1(n-1)}^\alpha,$$

- (iii) *The diagonal elements $a_{i(i+1),i(i+1)}^\alpha$, $1 \leq i \leq n-1$ are free to vary. The other diagonal elements satisfy*

$$a_{ik,ik}^\alpha = \sum_{p=i}^{k-1} a_{p(p+1),p(p+1)}^\alpha, \quad k > i + 1.$$

Lemma 2.7. [14] *The maximal number of non-nilpotent elements is*

$$f_{max} = n - 1.$$

3. MAIN RESULT

We denote by $L(n, f)$ a set of all non-nilpotent solvable Leibniz algebras with nilradical $T(n)$ and a complementary vector space $\langle X^1, X^2, \dots, X^f \rangle$.

Using notations (1) we have

$$R_{X^\alpha}(N) = A^\alpha N, \quad L_{X^\alpha}(N) = B^\alpha N,$$

where $A^\alpha = (a_{ij,pq}^\alpha)$, $B^\alpha = (b_{ij,pq}^\alpha)$, $1 \leq i < j \leq n$, $1 \leq p < q \leq n$.

Since the proof of the assertions concerning the elements of the matrix A^α in Lemma 2.6 uses only the property of derivation, one can check that it obviously extends to our case of Leibniz algebras. For the matrix B^α however, we have the next result.

Lemma 3.1. *The following relations hold:*

$$b_{ij,pq}^\alpha = -a_{ij,pq}^\alpha, \quad i + 1 < j, \quad (p, q) \neq (1, n)$$

Proof. From Lemma 2.6 we conclude

$$\begin{aligned} [N_{12}, X^\alpha] &= a_{12,12}^\alpha N_{12} + a_{12,2n}^\alpha N_{2n}, \\ [N_{i(i+1)}, X^\alpha] &= a_{i(i+1),i(i+1)}^\alpha N_{i(i+1)} + a_{i(i+1),1n}^\alpha N_{1n}, & 2 \leq i \leq n-2, \\ [N_{(n-1)n}, X^\alpha] &= a_{(n-1)n,(n-1)n}^\alpha N_{(n-1)n} + a_{(n-1)n,1(n-1)}^\alpha N_{1(n-1)}, \\ [N_{ij}, X^\alpha] &= \sum_{p=i}^{j-1} a_{p(p+1),p(p+1)}^\alpha N_{ij}, & i + 1 < j. \end{aligned}$$

It is easy to see that $[X^\alpha, N_{12}] + [N_{12}, X^\alpha]$ belongs to the right annihilator of the algebra of $L(n, f)$. From the chain of equalities

$$\begin{aligned} 0 &= [N_{12}, [X^\alpha, N_{12}] + [N_{12}, X^\alpha]] = [N_{12}, \sum_{i=3}^{n-1} b_{12,2i}^\alpha N_{2i} + (a_{12,2n}^\alpha + b_{12,2n}^\alpha) N_{2n}] = \\ &= \sum_{i=3}^{n-1} b_{12,2i}^\alpha N_{1i} + (a_{12,2n}^\alpha + b_{12,2n}^\alpha) N_{1n}, \end{aligned}$$

we deduce $b_{12,2j}^\alpha = 0$, $3 \leq j \leq n-1$ and $b_{12,2n}^\alpha = -a_{12,2n}^\alpha$.

Similarly, from

$$0 = [N_{1i}, [X^\alpha, N_{12}] + [N_{12}, X^\alpha]] = [N_{1i}, \sum_{j=i+1}^n b_{ij}^\alpha N_{ij}] = \sum_{j=i+1}^n b_{ij}^\alpha N_{1j}, \quad i > 2,$$

we derive $b_{12,ij}^\alpha = 0$, $2 < i < j \leq n$.

From the equality

$$0 = [N_{i(i+1)}, [X^\alpha, N_{12}] + [N_{12}, X^\alpha]], \quad i \geq 2,$$

we get

$$b_{12,12}^\alpha = -a_{12,12}^\alpha, \quad b_{12,1i}^\alpha = 0, \quad 3 \leq i \leq n-1.$$

Therefore, we obtain

$$[X^\alpha, N_{12}] = -a_{12,12}^\alpha N_{12} - a_{12,2n}^\alpha N_{2n} + b_{12,1n}^\alpha N_{1n}.$$

Applying analogous argumentations as we used above for the products with $k \geq 2$,

$$\begin{aligned} [N_{1k}, [X^\alpha, N_{i(i+1)}] + [N_{i(i+1)}, X^\alpha]], & \quad [N_{i(i+1)}, [X^\alpha, N_{i(i+1)}] + [N_{i(i+1)}, X^\alpha]], & 2 \leq i \leq n-2, \\ [N_{1k}, [X^\alpha, N_{(n-1)n}] + [N_{(n-1)n}, X^\alpha]], & \quad [N_{i(i+1)}, [X^\alpha, N_{(n-1)n}] + [N_{(n-1)n}, X^\alpha]], \\ [N_{1k}, [X^\alpha, N_{ij}] + [N_{ij}, X^\alpha]], & \quad [N_{i(i+1)}, [X^\alpha, N_{ij}] + [N_{ij}, X^\alpha]], & 1 < j-i < n-1, \\ [N_{1k}, [X^\alpha, N_{1n}] + [N_{1n}, X^\alpha]], & \quad [N_{i(i+1)}, [X^\alpha, N_{1n}] + [N_{1n}, X^\alpha]], \end{aligned}$$

we obtain

$$\begin{aligned} [X^\alpha, N_{i(i+1)}] &= -a_{i(i+1),i(i+1)}^\alpha N_{i(i+1)} + b_{i(i+1),1n}^\alpha N_{1n}, & 2 \leq i \leq n-2, \\ [X^\alpha, N_{(n-1)n}] &= -a_{(n-1)n,(n-1)n}^\alpha N_{(n-1)n} - a_{(n-1)n,1(n-1)}^\alpha N_{1(n-1)} + b_{(n-1)n,1n}^\alpha N_{1n}, \\ [X^\alpha, N_{ij}] &= -\sum_{p=i}^{j-1} a_{p(p+1),p(p+1)}^\alpha N_{ij} + b_{ij,1n}^\alpha N_{1n}, & 1 < j-i < n-1, \\ [X^\alpha, N_{1n}] &= b_{1n,1n}^\alpha N_{1n}. \end{aligned}$$

From the chain of equalities

$$\begin{aligned} [X^\alpha, N_{1n}] &= [X^\alpha, [N_{12}, N_{2n}]] = [[X^\alpha, N_{12}], N_{2n}] - [[X^\alpha, N_{2n}], N_{12}] = \\ &= [-a_{12,12}^\alpha N_{12} - a_{12,2n}^\alpha N_{2n} + b_{12,1n}^\alpha N_{1n}, N_{2n}] - [-\sum_{p=2}^{n-1} a_{p(p+1),p(p+1)}^\alpha N_{2n} + b_{2n,1n}^\alpha N_{1n}, N_{12}] = \\ &= -a_{12,12}^\alpha N_{1n} - \sum_{p=2}^{n-1} a_{p(p+1),p(p+1)}^\alpha N_{1n} = -\sum_{p=1}^{n-1} a_{p(p+1),p(p+1)}^\alpha N_{1n}, \end{aligned}$$

$$\text{we get } [X^\alpha, N_{1n}] = -\sum_{p=1}^{n-1} a_{p(p+1),p(p+1)}^\alpha N_{1n}.$$

By induction on j we will prove

$$(2) \quad [X^\alpha, N_{i(i+j)}] = -\sum_{p=i}^{i+j-1} a_{p(p+1),p(p+1)}^\alpha N_{i(i+j)}, \quad j-i \geq 2.$$

The base of induction ensures the equalities

$$\begin{aligned} [X^\alpha, N_{i(i+2)}] &= [X^\alpha, [N_{i(i+1)}, N_{(i+1)(i+2)}]] = [[X^\alpha, N_{i(i+1)}], N_{(i+1)(i+2)}] - \\ &= [[X^\alpha, N_{(i+1)(i+2)}], N_{i(i+1)}] = -\sum_{p=i}^{i+1} a_{p(p+1),p(p+1)}^\alpha N_{i(i+2)}, \quad 1 \leq i \leq n-2. \end{aligned}$$

Let us suppose that (2) holds for j and we will show it for $j+1$.

For $i+j+1 \leq n-1$ we have

$$\begin{aligned} [X^\alpha, N_{i(i+j+1)}] &= [X^\alpha, [N_{i(i+j)}, N_{(i+j)(i+j+1)}]] = [[X^\alpha, N_{i(i+j)}], N_{(i+j)(i+j+1)}] - \\ &= [[X^\alpha, N_{(i+j)(i+j+1)}], N_{i(i+j)}] = [-\sum_{p=i}^{i+j-1} a_{p(p+1),p(p+1)}^\alpha N_{i(i+j)}, N_{(i+j)(i+j+1)}] - \\ &= [-a_{(i+j)(i+j+1),(i+j)(i+j+1)}^\alpha N_{(i+j)(i+j+1)} + b_{(i+j)(i+j+1),1n}^\alpha N_{1n}, N_{i(i+j)}] = \\ &= -\sum_{p=i}^{i+j} a_{p(p+1),p(p+1)}^\alpha N_{i(i+j+1)}. \end{aligned}$$

The following chain of equalities complete the proof of equality (2)

$$\begin{aligned} [X^\alpha, N_{in}] &= [X^\alpha, [N_{i(n-1)}, N_{(n-1)n}]] = [X^\alpha, N_{i(n-1)}], N_{(n-1)n}] - [X^\alpha, N_{(n-1)n}], N_{i(n-1)}] = \\ &= [-\sum_{p=i}^{n-2} a_{p(p+1),p(p+1)}^\alpha N_{i(n-1)}, N_{(n-1)n}] - [-a_{(n-1)n,(n-1)n}^\alpha N_{(n-1)n} - a_{(n-1)n,1(n-1)}^\alpha N_{1(n-1)} + \\ & \quad b_{(n-1)n,1n}^\alpha N_{1n}, N_{i(n-1)}] = -\sum_{p=i}^{n-1} a_{p(p+1),p(p+1)}^\alpha N_{in}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} [X^\alpha, N_{12}] &= -a_{12,12}^\alpha N_{12} - a_{12,2n}^\alpha N_{2n} + b_{12,1n}^\alpha N_{1n}. \\ [X^\alpha, N_{i(i+1)}] &= -a_{i(i+1),i(i+1)}^\alpha N_{i(i+1)} + b_{i(i+1),1n}^\alpha N_{1n}, \quad 2 \leq i \leq n-2, \\ [X^\alpha, N_{(n-1)n}] &= -a_{(n-1)n,(n-1)n}^\alpha N_{(n-1)n} - a_{(n-1)n,1(n-1)}^\alpha N_{1(n-1)} + b_{(n-1)n,1n}^\alpha N_{1n}, \\ [X^\alpha, N_{ij}] &= -\sum_{p=i}^{j-1} a_{p(p+1),p(p+1)}^\alpha N_{ij}, \quad j > i+1. \end{aligned}$$

Comparison of the above products with notations in (1) completes the proof of lemma. \square

Lemma 3.2. For $1 \leq \alpha, \beta \leq n$ we have $[X^\alpha, X^\beta] = \sigma^{\alpha\beta} N_{1n}$ for some $\sigma^{\alpha\beta} \in \mathbb{C}$.

Proof. Consider

$$\begin{aligned} [N_{12}, [X^\alpha, X^\beta]] &= [[N_{12}, X^\alpha], X^\beta] - [[N_{12}, X^\beta], X^\alpha] = [a_{12,12}^\alpha N_{12} + a_{12,2n}^\alpha N_{2n}, X^\beta] - \\ &[a_{12,12}^\beta N_{12} + a_{12,2n}^\beta N_{2n}, X^\alpha] = a_{12,12}^\alpha (a_{12,12}^\beta N_{12} + a_{12,2n}^\beta N_{2n}) + a_{12,2n}^\alpha \left(\sum_{p=2}^{n-1} a_{p(p+1),p(p+1)}^\beta N_{2n} \right) - \\ &a_{12,12}^\beta (a_{12,12}^\alpha N_{12} + a_{12,2n}^\alpha N_{2n}) - a_{12,2n}^\beta \left(\sum_{p=2}^{n-1} a_{p(p+1),p(p+1)}^\alpha N_{2n} \right) = (a_{12,12}^\alpha a_{12,2n}^\beta - a_{12,12}^\beta a_{12,2n}^\alpha - \\ &\sum_{p=2}^{n-1} a_{p(p+1),p(p+1)}^\alpha a_{12,2n}^\beta + \sum_{p=2}^{n-1} a_{p(p+1),p(p+1)}^\beta a_{12,2n}^\alpha) N_{2n}. \end{aligned}$$

On the other hand,

$$[N_{12}, [X^\alpha, X^\beta]] = [N_{12}, \sum_{1 \leq q-p < n} \sigma_{pq}^{\alpha\beta} N_{pq}] = \sum_{i=3}^n \sigma_{2i}^{\alpha\beta} N_{1i}.$$

Comparing coefficients at the basis elements we derive

$$\sigma_{2i}^{\alpha\beta} = 0, \quad 3 \leq i \leq n.$$

For $2 \leq i \leq n-2$ we consider the chain of equalities

$$\begin{aligned} [N_{i(i+1)}, [X^\alpha, X^\beta]] &= [[N_{i(i+1)}, X^\alpha], X^\beta] - [[N_{i(i+1)}, X^\beta], X^\alpha] = \\ &a_{i(i+1),i(i+1)}^\alpha (a_{i(i+1),i(i+1)}^\beta N_{i(i+1)} + a_{i(i+1),1n}^\beta N_{1n}) + a_{i(i+1),1n}^\alpha \sum_{p=1}^{n-1} a_{p(p+1),p(p+1)}^\beta N_{1n} - \\ &a_{i(i+1),i(i+1)}^\beta (a_{i(i+1),i(i+1)}^\alpha N_{i(i+1)} + a_{i(i+1),1n}^\alpha N_{1n}) - a_{i(i+1),1n}^\beta \sum_{p=1}^{n-1} a_{p(p+1),p(p+1)}^\alpha N_{1n} = \\ &(a_{i(i+1),i(i+1)}^\alpha a_{i(i+1),1n}^\beta + a_{i(i+1),1n}^\alpha \sum_{p=1}^{n-1} a_{p(p+1),p(p+1)}^\beta - \\ &a_{i(i+1),i(i+1)}^\beta a_{i(i+1),1n}^\alpha - a_{i(i+1),1n}^\beta \sum_{p=1}^{n-1} a_{p(p+1),p(p+1)}^\alpha) N_{1n}. \end{aligned}$$

On the other hand,

$$[N_{i(i+1)}, [X^\alpha, X^\beta]] = [N_{i(i+1)}, \sum_{k=1}^{i-1} \sigma_{ki}^{\alpha\beta} N_{ki} + \sum_{j=i+2}^n \sigma_{(i+1)j}^{\alpha\beta} N_{(i+1)j}] = -\sum_{k=1}^{i-1} \sigma_{ki}^{\alpha\beta} N_{k(i+1)} + \sum_{j=i+2}^n \sigma_{(i+1)j}^{\alpha\beta} N_{ij}.$$

Therefore,

$$\sigma_{ki}^{\alpha\beta} = \sigma_{js}^{\alpha\beta} = 0, \quad 1 \leq k \leq i-1, \quad 2 \leq i \leq n-2, \quad 3 \leq j \leq n-1, \quad j+1 \leq s \leq n$$

and

$$[X^\alpha, X^\beta] = \sigma_{1(n-1)}^{\alpha\beta} N_{1(n-1)} + \sigma_{1n}^{\alpha\beta} N_{1n}.$$

Similar arguments for the products

$$[N_{(n-1)n}, [X^\alpha, X^\beta]]$$

yield $\sigma_{1(n-1)}^{\alpha\beta} = 0$, which completes the proof of the lemma. For convenience let us omit the lower indexes of $\sigma_{1n}^{\alpha\beta}$. \square

From Leibniz identity

$$[X^\alpha, [N_{i(i+1)}, X^\alpha]] = [[X^\alpha, N_{i(i+1)}], X^\alpha] - [[X^\alpha, X^\alpha], N_{i(i+1)}]$$

for $1 \leq i \leq n-1$ we obtain restrictions:

$$\begin{aligned} a_{i(i+1), i(i+1)}^\alpha (a_{i(i+1), 1n}^\alpha + b_{i(i+1), 1n}^\alpha) &= 0, \quad 2 \leq i \leq n-2, \\ a_{12, 12}^\alpha b_{12, 1n}^\alpha &= a_{(n-1)n, (n-1)n}^\alpha b_{(n-1)n, 1n}^\alpha = 0. \end{aligned}$$

Let us list again the obtained products between the basis elements. For $1 \leq \alpha \leq f$ we have

$$\left\{ \begin{array}{ll} [N_{12}, X^\alpha] = a_{12, 12}^\alpha N_{12} + a_{12, 2n}^\alpha N_{2n}, & \\ [N_{i(i+1)}, X^\alpha] = a_{i(i+1), i(i+1)}^\alpha N_{i(i+1)} + a_{i(i+1), 1n}^\alpha N_{1n}, & 2 \leq i \leq n-2, \\ [N_{(n-1)n}, X^\alpha] = a_{(n-1)n, (n-1)n}^\alpha N_{(n-1)n} + a_{(n-1)n, 1(n-1)}^\alpha N_{1(n-1)}, & \\ [N_{ij}, X^\alpha] = \sum_{p=i}^{j-1} a_{p(p+1), p(p+1)}^\alpha N_{ij}, & j > i+1, \\ [X^\alpha, N_{12}] = -a_{12, 12}^\alpha N_{12} - a_{12, 2n}^\alpha N_{2n} + b_{12, 1n}^\alpha N_{1n}, & \\ [X^\alpha, N_{i(i+1)}] = -a_{i(i+1), i(i+1)}^\alpha N_{i(i+1)} + b_{i(i+1), 1n}^\alpha N_{1n}, & 2 \leq i \leq n-2, \\ [X^\alpha, N_{(n-1)n}] = -a_{(n-1)n, (n-1)n}^\alpha N_{(n-1)n} - a_{(n-1)n, 1(n-1)}^\alpha N_{1(n-1)} + b_{(n-1)n, 1n}^\alpha N_{1n}, & \\ [X^\alpha, N_{ij}] = -\sum_{p=i}^{j-1} a_{p(p+1), p(p+1)}^\alpha N_{ij}, & j > i+1, \\ [X^\alpha, X^\beta] = \sigma^{\alpha\beta} N_{1n}, & \end{array} \right.$$

with restrictions on parameters:

$$\begin{aligned} a_{i(i+1), i(i+1)}^\alpha (a_{i(i+1), 1n}^\alpha + b_{i(i+1), 1n}^\alpha) &= 0, \quad 2 \leq i \leq n-2, \\ a_{12, 12}^\alpha b_{12, 1n}^\alpha &= a_{(n-1)n, (n-1)n}^\alpha b_{(n-1)n, 1n}^\alpha = 0. \end{aligned}$$

Note that for solvable non-Lie Leibniz algebras of the set $L(n, f)$ the following equality holds

$$(3) \quad [X^\gamma, N_{1n}] = [N_{1n}, X^\gamma] = 0, \quad 1 \leq \gamma \leq f.$$

Indeed, if we assume the contrary, then taking into account that $[X^\gamma, N_{1n}] = -[N_{1n}, X^\gamma]$ we can assume $[X^\gamma, N_{1n}] \neq 0$ for some $\gamma \in \{1, \dots, f\}$.

Simplifying the following products using Leibniz identity

$$\begin{aligned} &[X^\gamma, [N_{12}, X^\alpha] + [X^\alpha, N_{12}]], \quad [X^\gamma, [N_{i(i+1)}, X^\alpha] + [X^\alpha, N_{i(i+1)}]], \\ &[X^\gamma, [N_{(n-1)n}, X^\alpha] + [X^\alpha, N_{(n-1)n}]], \quad [X^\gamma, [X^\alpha, X^\beta] + [X^\beta, X^\alpha]], \quad [X^\gamma, [X^\alpha, X^\alpha]], \end{aligned}$$

we obtain

$$b_{12, 1n}^\alpha = b_{(n-1)n, 1n}^\alpha = \sigma^{\alpha\alpha} = 0, \quad b_{i(i+1), 1n}^\alpha = -a_{i(i+1), 1n}^\alpha, \quad \sigma^{\alpha\beta} = -\sigma^{\beta\alpha}.$$

Thus we get a Lie algebra, which is a contradiction.

Corollary 3.3. *For a Leibniz algebra of the set $L(n, 1)$ the matrices of the left and right operators $A = (a_{ij, pq})$, $B = (b_{ij, pq})$ have the following properties:*

- 1) *The maximum number of off-diagonal elements of matrix A is $n-1$;*
- 2) *The maximum number of off-diagonal elements of matrix B is $n+1$.*

Theorem 3.4. *Solvable Leibniz algebra of the set $L(n, n-1)$ is a Lie algebra.*

Proof. Making suitable change of basis we can assume that operator R_{X^1} acts as follows

$$\begin{aligned} [N_{12}, X^1] &= N_{12} + a_{12, 2n}^1 N_{2n}, \\ [N_{i(i+1)}, X^1] &= a_{i(i+1), 1n}^1 N_{1n}, & 2 \leq i \leq n-2, \\ [N_{(n-1)n}, X^1] &= a_{(n-1)n, 1(n-1)}^1 N_{1(n-1)}, \\ [N_{1j}, X^1] &= N_{1j}, & j > 2. \end{aligned}$$

Since $[N_{1n}, X^1] = N_{1n}$, then from Equation (3) it follows that the algebra is a Lie algebra. \square

So we present a description of solvable Leibniz algebras with nilradical $T(n)$. Moreover, in the case of maximal possible dimension we show that this algebra is a Lie algebra.

4. ILLUSTRATION FOR LOW DIMENSIONS

In this section we give the description of Leibniz algebras with nilradical $T(3)$ and $T(4)$.

Note that Lie algebra $T(3)$ is nothing else, but Heisenberg algebra $H(1)$. Solvable Leibniz algebras with Heisenberg nilradical were described in [10].

Therefore, we will consider case when $n = 4$. We know that the complimentary vector space to nilradical $T(4)$ has dimension less than four. In case when dimension of the complimentary space is equal to 3 we obtain a Lie algebra (see Theorem 3.4), which falls into the classification already obtained in [14]. So we will consider dimension of the complimentary vector space to be equal to 1 and 2.

The Leibniz algebras $L(4, 1)$.

From previous section we have that the algebra $L(4, 1)$ admits a basis $\{N_{12}, N_{23}, N_{34}, N_{13}, N_{24}, N_{14}, X\}$ in which the table of multiplication has the following form:

$$(4) \quad \begin{cases} [N_{12}, X] &= a_{12,12}N_{12} + a_{12,24}N_{24}, \\ [X, N_{12}] &= -a_{12,12}N_{12} - a_{12,24}N_{24} + b_{12,14}N_{14}, \\ [N_{23}, X] &= a_{23,23}N_{23} + a_{23,14}N_{14}, \\ [X, N_{23}] &= -a_{23,23}N_{23} + b_{23,14}N_{14}, \\ [N_{34}, X] &= -(a_{12,12} + a_{23,23})N_{34} + a_{34,13}N_{13}, \\ [X, N_{34}] &= (a_{12,12} + a_{23,23})N_{34} - a_{34,13}N_{13} + b_{34,14}N_{14}, \\ [N_{13}, X] &= -[X, N_{13}] = (a_{12,12} + a_{23,23})N_{13}, \\ [N_{24}, X] &= -[X, N_{24}] = -a_{12,12}N_{24}, \\ [X, X] &= \sigma_{14}N_{14}, \end{cases}$$

where

$$a_{12,12}b_{12,14} = a_{23,23}(a_{23,14} + b_{23,14}) = (a_{12,12} + a_{23,23})b_{34,14} = 0.$$

Since $L(4, 1)$ is a non-nilpotent Leibniz algebra we have $(a_{12,12}, a_{23,23}) \neq (0, 0)$.

Case 1. Let $a_{12,12} = 0$. Then $a_{23,23} \neq 0$, $b_{23,14} = -a_{23,14}$ and $b_{34,14} = 0$.

Taking the change of basis as follows:

$$X' = \frac{1}{a_{23,23}}X, \quad N'_{23} = N_{23} + \frac{a_{23,14}}{a_{23,23}}N_{14}, \quad N'_{34} = N_{34} - \frac{a_{34,13}}{2a_{23,23}}N_{13}$$

the multiplication (4) transforms into

$$\begin{aligned} [N_{12}, X] &= a_{12,24}N_{24}, & [X, N_{12}] &= -a_{12,24}N_{24} + b_{12,14}N_{14}, \\ [N_{23}, X] &= -[X, N_{23}] = N_{23}, & [N_{34}, X] &= -[X, N_{34}] = -N_{34}, \\ [N_{13}, X] &= -[X, N_{13}] = N_{13}, & [X, X] &= \sigma_{14}N_{14}, \end{aligned}$$

where $(b_{12,14}, \sigma_{14}) \neq (0, 0)$.

Case 2. Let $a_{12,12} \neq 0$, then $b_{12,14} = 0$. Taking the change of basis $X' = \frac{1}{a_{12,12}}X$, we can assume $a_{12,12} = 1$.

Subcase 2.1. Let $a_{23,23} = 0$. Then $b_{34,14} = 0$.

Applying a change of a basis

$$N'_{12} = N_{12} + \frac{a_{12,24}}{2}N_{24}, \quad N'_{34} = N_{34} - \frac{a_{34,13}}{2}N_{13}$$

the products (4) simplify to the following:

$$\begin{aligned} [N_{12}, X] &= -[X, N_{12}] = N_{12}, & [N_{34}, X] &= -[X, N_{34}] = -N_{34}, \\ [N_{13}, X] &= -[X, N_{13}] = N_{13}, & [N_{24}, X] &= -[X, N_{24}] = -N_{24}, \\ [N_{23}, X] &= a_{23,14}N_{14}, & [X, N_{23}] &= b_{23,14}N_{14}, \\ [X, X] &= \sigma_{14}N_{14}, \end{aligned}$$

where $(a_{23,14} + b_{23,14}, \sigma_{14}) \neq (0, 0)$.

Subcase 2.2. Let $a_{23,23} \neq 0$. Then $b_{23,14} = -a_{23,14}$.

Subcase 2.2.1. Let $a_{23,23} = -1$. Then substituting

$$N'_{23} = N_{23} - a_{23,14}N_{14}, \quad N'_{12} = N_{12} + \frac{a_{12,24}}{2}N_{24}$$

we derive to an algebra with the following table of multiplication:

$$\begin{aligned} [N_{12}, X] &= -[X, N_{12}] = N_{12}, & [N_{23}, X] &= [X, N_{23}] = -N_{23}, \\ [N_{34}, X] &= a_{34,13}N_{13}, & [X, N_{34}] &= -a_{34,13}N_{13} + b_{34,14}N_{14}, \\ [N_{24}, X] &= -[X, N_{24}] = -N_{24}, & [X, X] &= \sigma_{14}N_{14} \end{aligned}$$

where $(b_{12,14}, \sigma_{14}) \neq (0, 0)$.

Note that by permuting the indexes of the basis elements of the above algebra one obtains an algebra from Case 1.

Subcase 2.2.2. Let $a_{23,23} \neq -1$. Then $b_{34,14} = 0$.

Setting

$$\begin{aligned} N'_{12} &= N_{12} + \frac{a_{12,24}}{2}N_{24}, & N'_{23} &= N_{23} + \frac{a_{23,14}}{a_{23,23}}N_{14}, \\ N'_{34} &= \sigma_{14}(N_{34} - \frac{a_{34,13}}{2(1+a_{23,23})}N_{13}), & N'_{24} &= \sigma_{14}N_{24}, & N'_{14} &= \sigma_{14}N_{14} \end{aligned}$$

we get an algebra with the following table of multiplications:

$$\begin{aligned} [N_{12}, X] &= -[X, N_{12}] = N_{12}, & [N_{23}, X] &= -[X, N_{23}] = a_{23,23}N_{23}, \\ [N_{34}, X] &= -[X, N_{34}] = -(1+a_{23,23})N_{34}, & [N_{13}, X] &= -[X, N_{13}] = (1+a_{23,23})N_{13}, \\ [N_{24}, X] &= -[X, N_{24}] = -N_{24}, & [X, X] &= N_{14}, \end{aligned}$$

where $(1+a_{23,23})a_{23,23} \neq 0$.

Non-isomorphisms of obtained algebras can be easily established by considering the dimensions of derived series of the algebras.

Thus, the following theorem is proved.

Theorem 4.1. *An arbitrary non-Lie Leibniz algebra of the set $L(4, 1)$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{aligned} L_1 : \quad [N_{12}, X] &= a_{12,24}N_{24}, & [X, N_{12}] &= -a_{12,24}N_{24} + b_{12,14}N_{14}, \\ [N_{23}, X] &= -[X, N_{23}] = N_{23}, & [N_{34}, X] &= -[X, N_{34}] = -N_{34}, \\ [N_{13}, X] &= -[X, N_{13}] = N_{13}, & [X, X] &= \sigma_{14}N_{14}, \end{aligned}$$

where $(b_{12,14}, \sigma_{14}) \neq (0, 0)$.

$$\begin{aligned} L_2 : \quad [N_{12}, X] &= -[X, N_{12}] = N_{12}, & [N_{34}, X] &= -[X, N_{34}] = -N_{34}, \\ [N_{13}, X] &= -[X, N_{13}] = N_{13}, & [N_{24}, X] &= -[X, N_{24}] = -N_{24}, \\ [N_{23}, X] &= a_{23,14}N_{14}, & [X, N_{23}] &= b_{23,14}N_{14}, \\ [X, X] &= \sigma_{14}N_{14}, \end{aligned}$$

where $(a_{23,14} + b_{23,14}, \sigma_{14}) \neq (0, 0)$.

$$\begin{aligned} L_3 : \quad [N_{12}, X] &= -[X, N_{12}] = N_{12}, & [N_{23}, X] &= -[X, N_{23}] = a_{23,23}N_{23}, \\ [N_{34}, X] &= -[X, N_{34}] = -(1+a_{23,23})N_{34}, & [N_{13}, X] &= -[X, N_{13}] = (1+a_{23,23})N_{13}, \\ [N_{24}, X] &= -[X, N_{24}] = -N_{24}, & [X, X] &= N_{14}. \end{aligned}$$

where $(1+a_{23,23})a_{23,23} \neq 0$.

The Leibniz algebras $L(4, 2)$.

Classification of Leibniz algebras in this set is presented in the following theorem.

Theorem 4.2. *An arbitrary non-Lie Leibniz algebra of the set $L(4, 2)$ admits a basis $\{N_{12}, N_{23}, N_{34}, N_{13}, N_{24}, N_{14}, X^1, X^2\}$ in which the table of multiplication has the following form:*

$$\begin{aligned} [N_{12}, X^1] &= -[X^1, N_{12}] = N_{12}, & [N_{34}, X^1] &= -[X^1, N_{34}] = -N_{34}, \\ [N_{13}, X^1] &= -[X^1, N_{13}] = N_{13}, & [N_{24}, X^1] &= -[X^1, N_{24}] = -N_{24}, \\ [N_{23}, X^2] &= -[X^2, N_{23}] = N_{23}, & [N_{34}, X^2] &= -[X^2, N_{34}] = -N_{34}, \\ [N_{13}, X^2] &= -[X^2, N_{13}] = N_{13}, & [X^1, X^1] &= \sigma^{11}N_{14}, \\ [X^2, X^2] &= \sigma^{22}N_{14}, & [X^1, X^2] &= \sigma^{12}N_{14}, & [X^2, X^1] &= \sigma^{21}N_{14}. \end{aligned}$$

Proof. From Lemmas 3.1 and 3.2 we have

$$\begin{aligned}
[N_{12}, X^1] &= a_{12,12}^1 N_{12} + a_{12,24}^1 N_{24}, \\
[X^1, N_{12}] &= -a_{12,12}^1 N_{12} - a_{12,24}^1 N_{24} + b_{12,14}^1 N_{14}, \\
[N_{23}, X^1] &= a_{23,23}^1 N_{23} + a_{23,14}^1 N_{14}, \\
[X^1, N_{23}] &= -a_{23,23}^1 N_{23} + b_{23,14}^1 N_{14}, \\
[N_{34}, X^1] &= -(a_{12,12}^1 + a_{23,23}^1) N_{34} + a_{34,13}^1 N_{13}, \\
[X^1, N_{34}] &= (a_{12,12}^1 + a_{23,23}^1) N_{34} - a_{34,13}^1 N_{13} + b_{34,14}^1 N_{14}, \\
[N_{13}, X^1] &= -[X^1, N_{13}] = (a_{12,12}^1 + a_{23,23}^1) N_{13}, \\
[N_{24}, X^1] &= -[X^1, N_{24}] = -a_{12,12}^1 N_{24}, \\
[N_{12}, X^2] &= a_{12,12}^2 N_{12} + a_{12,24}^2 N_{24}, \\
[X^2, N_{12}] &= -a_{12,12}^2 N_{12} - a_{12,24}^2 N_{24} + b_{12,14}^2 N_{14}, \\
[N_{23}, X^2] &= a_{23,23}^2 N_{23} + a_{23,14}^2 N_{14}, \\
[X^2, N_{23}] &= -a_{23,23}^2 N_{23} + b_{23,14}^2 N_{14}, \\
[N_{34}, X^2] &= -(a_{12,12}^2 + a_{23,23}^2) N_{34} + a_{34,13}^2 N_{13}, \\
[X^2, N_{34}] &= (a_{12,12}^2 + a_{23,23}^2) N_{34} - a_{34,13}^2 N_{13} + b_{34,14}^2 N_{14}, \\
[N_{13}, X^2] &= -[X^2, N_{13}] = (a_{12,12}^2 + a_{23,23}^2) N_{13}, \\
[N_{24}, X^2] &= -[X^2, N_{24}] = -a_{12,12}^2 N_{24}
\end{aligned}$$

with the restrictions

$$a_{12,12}^1 b_{12,14}^1 = a_{23,23}^1 (a_{23,14}^1 + b_{23,14}^1) = (a_{12,12}^1 + a_{23,23}^1) b_{34,14}^1 = 0,$$

$$a_{12,12}^2 b_{12,14}^2 = a_{23,23}^2 (a_{23,14}^2 + b_{23,14}^2) = (a_{12,12}^2 + a_{23,23}^2) b_{34,14}^2 = 0.$$

Taking the change of basis

$$\begin{aligned}
X^{1'} &= \frac{a_{23,23}^2}{a_{12,12}^1 a_{23,23}^2 - a_{12,12}^2 a_{23,23}^1} X^1 - \frac{a_{23,23}^1}{a_{12,12}^1 a_{23,23}^2 - a_{12,12}^2 a_{23,23}^1} X^2, \\
X^{2'} &= -\frac{a_{12,12}^2}{a_{12,12}^1 a_{23,23}^2 - a_{12,12}^2 a_{23,23}^1} X^1 + \frac{a_{12,12}^1}{a_{12,12}^1 a_{23,23}^2 - a_{12,12}^2 a_{23,23}^1} X^2,
\end{aligned}$$

we deduce

$$\begin{aligned}
[N_{12}, X^1] &= -[X^1, N_{12}] = N_{12} + a_{12,24}^1 N_{24}, & [N_{23}, X^1] &= a_{23,14}^1 N_{14}, \\
[X^1, N_{23}] &= b_{23,14}^1 N_{14}, & [N_{34}, X^1] &= -[X^1, N_{34}] = -N_{34} + a_{34,13}^1 N_{13}, \\
[N_{13}, X^1] &= -[X^1, N_{13}] = N_{13}, & [N_{24}, X^1] &= -[X^1, N_{24}] = -N_{24}, \\
[N_{12}, X^2] &= a_{12,24}^2 N_{24}, & [X^2, N_{12}] &= -a_{12,24}^2 N_{24} + b_{12,14}^2 N_{14}, \\
[N_{23}, X^2] &= -[X^2, N_{23}] = N_{23} + a_{23,14}^2 N_{14}, & [N_{34}, X^2] &= -[X^2, N_{34}] = -N_{34} + a_{34,13}^2 N_{13}, \\
[N_{13}, X^2] &= -[X^2, N_{13}] = N_{13}.
\end{aligned}$$

Applying Leibniz identity for the following triples of elements:

$$(N_{12}, X^1, X^2), (N_{23}, X^1, X^2), (N_{34}, X^1, X^2), (X^1, N_{23}, X^2), (X^2, N_{12}, X^1)$$

we get

$$a_{12,24}^2 = a_{23,14}^1 = a_{34,13}^1 = a_{34,13}^2 = b_{23,14}^1 = b_{12,14}^2 = 0.$$

Finally, taking the basis transformation:

$$N'_{12} = N_{12} + \frac{a_{12,24}^1}{2} N_{24}, \quad N'_{23} = N_{23} + a_{23,14}^2 N_{14}$$

we obtain the table of multiplication listed in the assertion of theorem. \square

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